

Zero entropy subgroups of mapping class groups

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Abstract

Given a group action on a surface with a finite invariant set we investigate how the algebraic properties of the induced group of permutations of that set affects the dynamical properties of the group. Our main result shows that in many circumstances if the induced permutation group is not solvable then among the homeomorphisms in the group there must be one with a pseudo-Anosov component. We formulate this in terms of the mapping class group relative to the finite set and show the stronger result that in many circumstances (e.g. if the surface has boundary) if this mapping class group has no elements with pseudo-Anosov components then it is itself solvable.

1 Introduction

Let M be a compact surface with boundary. We are interested in the question of how a group action on M permutes a finite invariant set $X \subset \text{int}(M)$. More precisely, how the algebraic properties of the induced group of permutations of a finite invariant set affects the dynamical properties of the group. Our main result shows that in many circumstances if the induced permutation group is not solvable then among the homeomorphisms in the group there must be one with a pseudo-Anosov component. We formulate this in terms of the mapping class group relative to the finite set and show the stronger result that in many circumstances (e.g. if $\partial M \neq \emptyset$) this mapping class group is itself solvable if it has no elements with pseudo-Anosov components.

Definition 1.1. By $\mathcal{B}(M, X)$ we denote the mapping class group of $M \setminus X$, where X is a finite, non-empty set. In the notation of [3], if $S = M \setminus X$ then $\mathcal{B}(M, X) = \text{Mod}(S, \partial S)$.

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So an element of $\mathcal{B}(M, X)$ is an isotopy class of homeomorphisms of M which fix ∂M pointwise and permute the elements of X . The isotopies as well as the homeomorphisms are required to fix ∂M pointwise.

We will say that an element g of $\mathcal{B}(M, X)$ has *zero entropy* if it has a representative with zero entropy. An equivalent and very useful way to say this is that g has a Thurston canonical form with no irreducible components of pseudo-Anosov type (see Theorem 4.1 below). We will say that a subgroup G of $\mathcal{B}(M, X)$ has *zero entropy* if each element has entropy zero. For any subgroup G of $\mathcal{B}(M, X)$ there is a natural homomorphism $\pi : G \rightarrow \mathcal{S}_G(X)$ where $\mathcal{S}_G(X)$ is the group of permutations of X induced by G . We will denote the derived length of a solvable group G by $\text{DLen}(G)$.

We first consider the case of surfaces with genus zero.

Theorem 1.2. *Suppose M is an oriented compact connected surface of genus 0 and let G be a finitely generated infinite subgroup of $\mathcal{B}(M, X)$ with zero entropy, where X is a non-empty finite set. If $\partial M \neq \emptyset$, or there is a point $x \in X$ fixed by all elements of G , then G is solvable. If $M = S^2$ then there is a finite index normal subgroup G_0 of G which is solvable and such that G/G_0 acts effectively on S^2 . Moreover when G is solvable the kernel K of the homomorphism $\pi : G \rightarrow \mathcal{S}_G(X)$ is free abelian.*

Corollary 1.3. *Suppose G satisfies the hypothesis of Theorem 1.2. If G is solvable then its derived length $\text{DLen}(G)$ satisfies*

$$\text{DLen}(G) - 1 \leq \text{DLen}(\mathcal{S}_G(X)) \leq \text{DLen}(G).$$

Remark 1.4. Note that as a consequence if G satisfies the hypothesis of Theorem 1.2, it is virtually abelian. This follows by applying Theorem 1.2 to the solvable group G_0 and noting G_0 has finite index in G and G_0/K is finite. This is also a consequence of Theorem B of [1] and results presented in [6].

An interesting question asks which permutations of a finite set force a braid inducing that permutation on its ends to have a pseudo-Anosov component. An interesting corollary of Theorem 1.2 gives a necessary algebraic condition, not for a single permutation, but for a subgroup of the permutation group S_n induced by a subgroup of the braid group on n -strands.

Corollary 1.5. *Suppose G is a finitely generated subgroup of the braid group on n -strands and $\pi : G \rightarrow S_n$ is the natural projection onto the permutation group of the ends of the strands. If G has entropy zero then it is solvable and the kernel of π is free abelian so G is virtually abelian and has derived length satisfying $\text{DLen}(G) - 1 \leq \text{DLen}(\pi(G)) \leq \text{DLen}(G)$.*

When the genus of the surface M is positive we have the following result. As in Theorem 1.2 we assume X is a non-empty finite set.

Theorem 1.6. *Suppose M is an oriented compact connected surface of genus $g > 1$ and let G be a finitely generated infinite subgroup of $\mathcal{B}(M, X)$ which contains no element with positive entropy.*

1. *If $\partial M \neq \emptyset$, or there exists $x \in X$ fixed by each element of G , then G is solvable.*
2. *Otherwise G contains a solvable subgroup G_0 of index at most $84(g-1)(2g-2)!$ in G .*

There is also a result for the case of the torus. We continue with the assumption that X is a non-empty finite set.

Theorem 1.7. *Suppose M is an oriented compact connected surface of genus $g = 1$ and let G be a finitely generated subgroup of $\mathcal{B}(M, X)$ which contains no element with positive entropy. Then G is solvable.*

2 Graph Maps

We will have need of an elementary result about groups acting on a finite tree.

Proposition 2.1. *Let G be a group acting isometrically on a finite tree. Then $\text{Fix}(G) \neq \emptyset$.*

Proof. Let T be a finite tree. If e is an edge of T , which has been assigned an orientation, we will denote by $T^+(e)$ and $T^-(e)$ the two subtrees whose union is the complement of the interior of e . The subtree $T^+(e)$ is chosen to contain the positive endpoint of e and $T^-(e)$ the negative.

We wish to orient the edges of T so that for every edge e the subtree $T^+(e)$ contains more than half of the vertices of T . This is possible unless there is an edge e_0 such that $T^+(e_0)$ and $T^-(e_0)$ each contain exactly half the vertices of T . Such an edge is unique, if it exists, because every other edge lies in (and separates) either $T^+(e_0)$ or $T^-(e_0)$.

Since the edge e_0 is unique, if it exists, it must be preserved by each element of G and hence the midpoint of e_0 is an element of $\text{Fix}(G)$.

We are left with the case that every edge of T can be oriented in such a way that the subtree $T^+(e)$ contains more than half of the vertices of T . Note that this implies that at most one edge can exit from any vertex. If one starts at any point on the tree and follows a path consistent with the orientation this path will necessarily end in a sink vertex v_0 (i.e. one with no exiting edges) since no cycles are possible.

We claim that this sink is unique. This is because its “basin of attraction,” i.e. the union of all edges on paths which lead to v_0 , is a subtree of T . If there were more than one such subtree two would intersect and their intersection would contain a vertex with more than one exiting edge.

Since v_0 is unique it must be a fixed point for every element of G . □

3 Dehn Twists

We denote \mathbb{R}/\mathbb{Z} by \mathbb{T}^1 .

Definition 3.1. A generalized Dehn twist between two copies of $\mathbb{A} = \mathbb{T}^1 \times I$, say \mathbb{A}_0 and \mathbb{A}_1 is a homeomorphism $\alpha : \mathbb{A}_0 \rightarrow \mathbb{A}_1$ which for some $a, b \in \mathbb{Q}$ is given by $\alpha(x, t) = (x + at + b \pmod{1}, t)$. A generalized Dehn twist is called essential if $a \neq 0$.

We will frequently be interested in the case that $\mathbb{A}_0 = \mathbb{A}_1$. Observe that in this case the generalized Dehn twists on $\mathbb{A} = \mathbb{A}_0 = \mathbb{A}_1$ form an abelian group. Note that since $a, b \in \mathbb{Q}$, $\alpha : \mathbb{A} \rightarrow \mathbb{A}$ has finite order on the boundary of $S^1 \times I$.

Lemma 3.2. Suppose $f : \mathbb{A}_0 \rightarrow \mathbb{A}_1$ is a homeomorphism which is a rational rotation on each component of $\partial\mathbb{A}_0$. Then there is a unique generalized Dehn twist $\alpha(f)$ on \mathbb{A}_0 which is isotopic to f relative to $\partial\mathbb{A}_0$. If g is another such homeomorphism then $\alpha(f \circ g) = \alpha(f) \circ \alpha(g) = \alpha(g) \circ \alpha(f) = \alpha(g \circ f)$.

Proof. Let $\tilde{\mathbb{A}} = \mathbb{R} \times I$ be the universal cover of \mathbb{A} . Choose $b_0 \in \mathbb{R}$ which projects to the rotation $f|_{\mathbb{T} \times \{0\}}$ on \mathbb{T} . Let $\tilde{F} : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ be the lift for which $\tilde{f}|_{\mathbb{R} \times \{0\}}$ is translation by b_0 . Then $\tilde{f}|_{\mathbb{R} \times \{1\}}$ is translation by some $b_1 \in \mathbb{R}$. We define $\tilde{\alpha} : \tilde{\mathbb{A}} \rightarrow \tilde{\mathbb{A}}$ by $\tilde{\alpha}(x, t) = x + at + b_0$ where $a = b_1 - b_0$. The homeomorphism $\tilde{\alpha}$ is equivariant with respect to the \mathbb{Z} action on $\tilde{\mathbb{A}}$ so $\tilde{\alpha}$ is the lift of a generalized Dehn twist $\alpha(f)$. If g is another homeomorphism with the properties of f then from the construction it is clear that $\alpha(f \circ g) = \alpha(f) \circ \alpha(g)$. Since the group of generalized Dehn twists is abelian it follows that $\alpha(f \circ g) = \alpha(g \circ f)$. \square

4 Thurston's Theorem

We cite a special case of Thurston's theorem on classification of surface homeomorphisms up to isotopy. We limit ourselves to those homeomorphisms with zero entropy or equivalently have no pseudo-Anosov components as this is appropriate for our needs. Throughout this section $\text{Homeo}_0(M)$ denotes the orientation preserving homeomorphisms of the compact surface M .

Theorem 4.1 (Thurston). Suppose M is a compact oriented surface and $X \subset M$ is finite. Let $[g]$ be an element of $\mathcal{B}(M, X)$ which has a representative with entropy zero. There exists a homeomorphism f representing $[g]$ and a finite set of pairwise disjoint closed annuli $\mathcal{U} = \{A_j\}$ in $M \setminus X$ with the following properties:

1. For each component S of ∂M there is an annulus $A \in \mathcal{U}$ such that S is one of the boundary components of A .
2. The diffeomorphism f permutes the elements of $\mathcal{U} = \{A_i\}$

3. The restriction of f to each A_i is a generalized Dehn twist from A_i to its image. This twist is essential if A_i does not contain a component of ∂M .
4. If Y is a component of the complement of $U = \bigcup_i A_i$ which is invariant under f^k then $f^k|_Y$ has finite order.

Remark 4.2. The isotopy class $[c_i]$ of an essential simple closed curve c_i in A_i which is not isotopic to a component of ∂M is called an *essential reducing curve*. The curves c_i and the annuli A_i are unique up to isotopy (see [1]). Each of the neighborhoods A_i may be chosen to lie in an arbitrarily small neighborhood of a component of ∂M or of an essential reducing curve c_i .

We will say that two embedded annuli A_1 and A_2 *intersect essentially* in $M \setminus X$ if any embedded simple closed curves c_1 and c_2 , which are isotopic to core curves of A_1 and A_2 respectively, must have a point of intersection. We will say they *intersect minimally* if there intersection has the fewest number of components among all choices of isotopy class representatives for A_1 and A_2 .

Lemma 4.3. *Let M be a compact oriented surface and let $X \subset M$ be a finite subset. Suppose $[g_1], [g_2] \in \mathcal{B}(M, X)$ each satisfy the hypothesis of Theorem 4.1, i.e., they contain zero entropy representatives. For $i = 1, 2$ let f_i denote the homeomorphism guaranteed by this Theorem and let \mathcal{U}_i denote the associated family of annuli. Then if $U_i(g_1) \in \mathcal{U}(g_1)$ and $U_j(g_2) \in \mathcal{U}(g_2)$ intersect essentially, the group \mathcal{G} generated by $[f_1] = [g_1]$ and $[f_2] = [g_2]$ in $\mathcal{B}(M, X)$ has elements each of whose representatives has positive entropy.*

Proof. Since $[f_i] = [g_i]$ we may without loss of generality choose $g_i = f_i$. By taking powers if necessary we may assume g_i preserves the components of $\mathcal{U}(g_i)$. Theorem 4.6 of [8] (see also [4] for the result as stated below) implies that any subgroup G of $\mathcal{B}(M, X)$ satisfies one of the following:

1. G contains a pair of independent pseudo-Anosov elements (such subgroups are called sufficiently large in [8]),
2. G fixes the laminations $\{F^-(g); F^+(g)\}$ for a certain pseudo-Anosov element g ,
3. G is an infinite subgroup fixing (up to isotopy) a finite disjoint system of non-peripheral essential simple closed curves on M (such subgroups are called reducible),
4. G is finite.

If G is the subgroup generated by $[g_1]$ and $[g_2]$ then it contains elements of infinite order (e.g. $[g_i]$) so (4) is impossible. In cases (1) and (2) the group G contains

elements which are isotopic to pseudo-Anosov maps and hence any representative of such an element has positive entropy.

We are left with case (3), the reducible case. In this case we claim that no element of the finite set C of non-peripheral essential simple closed curves invariant under G can have an essential intersection with an element of $\mathcal{U}(g_i)$.

To show this claim we choose a hyperbolic metric on M and geodesic representatives $\{c_j\}$ of the elements of C and also choose geodesic representatives $\{u_k\}$ of $\mathcal{U}(g_i)$. We assume that c_0 and $u_0 \in \mathcal{U}(g_1)$ have an essential intersection and show this leads to a contradiction, the other cases being similar. Let h be isotopic to a power of g_1 which twists more than once around u_0 and is chosen so that $h = id$ outside of the annuli $\mathcal{U}(g_1)$. Let \tilde{u}_0 be a lift of u_0 to the universal cover \tilde{M} and let $\tilde{U}_0(g_1)$ be the lift of $U_0(g_1)$ containing \tilde{u}_0 . Since u_0 and c_0 intersect essentially there is a lift \tilde{c}_0 which intersects \tilde{u}_0 . There are lifts of h which fix the endpoints p^+, p^- of \tilde{u}_0 in the compactification of \tilde{M} obtained by adding the circle at infinity S_∞ . These lifts all differ by composition with a power of T , the covering translation corresponding to c_0 . Because h twists more than one turn there is a lift \tilde{h} of h which preserves $\tilde{U}_0(g_1)$ and moves points in the boundary components of $\tilde{U}_0(g_1)$ in opposite directions, i.e. for z in one component of $\partial\tilde{U}_0(g_1)$ we have $\tilde{h}(z) = T^d(z)$ for some $d > 0$ while for z' in the other component $\tilde{h}(z') = T^{d'}(z')$ for some $d' < 0$.

It follows that \tilde{h} fixes p^+ and p^- and moves the points in the components of $S_\infty \setminus \{p^+, p^-\}$ in opposite directions. More precisely, we will show, that for some x_0 in one component of $S_\infty \setminus \{p^+, p^-\}$ we have $\tilde{h}^n(x_0) = T^{nd}(x_0)$ and hence for any x in the same component of $S_\infty \setminus \{p^+, p^-\}$ will have

$$\omega(x, \tilde{h}) = \lim_{n \rightarrow \infty} \tilde{h}^n(x_0) = \lim_{n \rightarrow \infty} T^{nd}(x_0) = p^+,$$

and similarly for x in the other component we have $\omega(x, \tilde{h}) = p^-$. To see this consider a curve $\tilde{\gamma}(t)$ in \tilde{M} starting at a point $\gamma(0) = z \in \partial\tilde{U}_0(g_1)$ and ending at a point of $S_\infty \setminus \{p^+, p^-\}$ but with $\gamma(t) \notin \tilde{U}_i(g_1)$ for all i and all $t > 0$. Then the projection of $\gamma(t)$ to M lies in the fixed point set of h , so the lift \tilde{h} must take $\tilde{\gamma}(t)$ to $T_0(\tilde{\gamma}(t))$ for some covering translation T_0 independent of t . Since $\tilde{h}(z) = T^d(z)$ it must be that $T_0 = T^d$ and $\tilde{h}^n(x_0) = T^{nd}(x_0)$ where x_0 is the end of $\gamma(t)$ in S_∞ .

To produce the contradiction giving us the claim we first note that since the geodesics c_0 and u_0 intersect essentially there is a lift \tilde{c}_0 of c_0 which intersects \tilde{u}_0 . The ends of \tilde{c}_0 are then in different components of $S_\infty \setminus \{p^+, p^-\}$. We know that $h(c_0)$ is isotopic to c_0 . Hence the lifted geodesic \tilde{c}' whose ends are the ends of $\tilde{h}(\tilde{c}_0)$ is another lift of c_0 and there must be a covering translation T_1 such that $T_1(\tilde{c}_0) = \tilde{c}'$. But since \tilde{h} moves ends of \tilde{c}_0 in opposite directions the ends of \tilde{c}' are separated in $\tilde{M} \cup S_\infty$ by \tilde{c}_0 . This would require that \tilde{c}' and \tilde{c}_0 , two lifts of c_0 , intersect. This contradiction proves the claim that no element of the finite set C of non-peripheral essential simple closed curves invariant under G can have an essential intersection with an element of $\mathcal{U}(g_i)$.

We can now complete the proof of the lemma. Since all elements of C have representatives disjoint from $\mathcal{U}(g_1)$ and $\mathcal{U}(g_2)$, we may reduce along the curves of C obtaining an induced subgroup of the mapping class of their complement. One component P of this complement contains elements of both $\mathcal{U}(g_1)$ and $\mathcal{U}(g_2)$ which intersect essentially. This component must be invariant under $[g'_1]$ and $[g'_2]$ where $[g'_i]$ is the element of the mapping class of $P \setminus X$ induced by $[g_i]$. Hence we may consider the maps g'_1 and g'_2 and the subgroup G_P of the mapping class group of $P \setminus X$ which they generate. We again must have one of the four alternatives listed above. Case (4) is not possible since g_i has infinite order. Case (3) is not possible since we have reduced as much as possible. Cases (1) and (2) imply there is an element $[g] \in G$ such that mapping class of $P \setminus X$ which it induces is pseudo-Anosov. It follows that $[g]$ has a pseudo-Anosov component and in particular any g representing it has positive entropy. \square

Lemma 4.4. *Suppose M is a compact oriented surface, $X \subset \text{int}(M)$ is a non-empty finite subset, and $M \setminus X$ has negative Euler characteristic and is provided with a hyperbolic metric. Suppose also that G is a finitely generated subgroup $\mathcal{B}(M, X)$ every element of which has finite order. Then there is a finite group \mathcal{G} of isometries of $M \setminus X$ and an isomorphism $\psi : G \rightarrow \mathcal{G}$ such that for each $[h] \in G$ the homeomorphism $\psi([h])$ is in the isotopy class $[h]$.*

Proof. This is almost a result of Kerckhoff (see [7] or Theorem 7.2 of [3]), but we need to know that G is a finite group. For this we follow a proof given in Chapter 7 of [3]. The action of G on $H_1(M \setminus X, \mathbb{R})$ is linear and hence its image is finite by the classical Burnside result. Hence G has a finite index subgroup G_0 which acts as the identity on H_1 . If $[h] \in G_0$ then it has finite order and can be realized by a finite order isometry of $M \setminus X$ (this follows, e.g., from Kerckhoff's theorem. See [7] or Theorem 7.2 of [3]). We would like to apply the Lefschetz fixed point theorem which requires a compact space, so we let M_0 be the complement in M of small open h -invariant neighborhoods of points of X . The action of h is trivial on $H_1(M_0)$. Since h is an isometry of M_0 , if $h \neq id$, then its fixed points are isolated and each fixed point has Lefschetz index 1. It follows that $\text{Card}(\text{Fix}(f)) = L(f)$ where $L(f)$ is the Lefschetz number of f . By the Lefschetz fixed point theorem $L(f) = \text{tr}(f_{*0}) - \text{tr}(f_{*1}) = 1 - \dim(H_1(M_0))$ (because $\text{tr}(f_{*2}) = 0$). Since M_0 has negative Euler characteristic $\dim(H_1(M_0)) \geq 2$ which implies $L(f) < 0$ so $L(f) \neq \text{Card}(\text{Fix}(f))$. This contradiction implies $h = id$. We conclude that G_0 is trivial and G is finite.

Now by Kerckhoff's theorem (see [7] or Theorem 7.2 of [3]) G can be realized by a group \mathcal{G} of isometries of a hyperbolic metric. \square

Lemma 4.5. *Suppose M is a compact oriented surface and $X \subset M$ is finite. Let G be a finitely generated subgroup of $\mathcal{B}(M, X)$ each element of which has a representative of zero entropy. There exists a set \mathcal{C} of simple closed curves in $M \setminus X$ such that*

1. *Each $c \in \mathcal{C}$ is a representative of an isotopy class of an essential reducing*

curve for some element of G and each such isotopy class is represented by some element of \mathcal{C} .

2. Elements of \mathcal{C} are pairwise disjoint.
3. The set \mathcal{C} is invariant up to isotopy in $M \setminus X$ under G .
4. If $\mathcal{C} \neq \emptyset$ each component of the complement of \mathcal{C} in $M \setminus X$ has negative Euler characteristic.
5. \mathcal{C} is finite.

Proof. Let $\bar{\mathcal{C}} = \{[c_j]\}$ be the set of isotopy classes of all essential reducing curves for all elements of G acting on $M \setminus X$. We may choose the representative c_i of $[c_i]$ so that $c_i \cap c_j = \emptyset$ if $i \neq j$ since otherwise there would be an element with positive entropy by Lemma 4.3. Let $\mathcal{C} = \{c_j\}$ then (1) and (2) are satisfied.

The set \mathcal{C} is invariant up to isotopy under G since if c_i is a reducing curve for f then $g(c_i)$ is a reducing curve for gfg^{-1} . This proves (3).

If $\mathcal{C} \neq \emptyset$ and Y is a component of the complement of \mathcal{C} in $M \setminus X$ with non-negative Euler characteristic then Y is homeomorphic to the annulus or the disk. If it were the annulus the two boundary components of Y would be isotopic in $M \setminus X$ and hence represent the same element of \mathcal{C} . If it were a disk its boundary would be a null-homotopic reducing curve for some element of G which does not occur by definition. Hence we have shown (4).

The set \mathcal{C} must be finite since if the complement of \mathcal{C} has more than n components each with negative Euler characteristic, then the Euler characteristic of $M \setminus X$ is at most $-n$. But this Euler characteristic is $\chi(M) - \text{Card}(X)$ and hence one calculates that $n \leq \text{Card}(X) - \chi(M) = \text{Card}(X) + 2g - 2$ where g is the genus of M . This proves (5). \square

The following result is implicit in Theorem 4.6 of [8]. Since it largely follows from results of Thurston and Kerckhoff, we give the proof here.

Lemma 4.6. *Suppose M is a compact oriented surface and $X \subset M$ is finite. Let G be a finitely generated subgroup of $\mathcal{B}(M, X)$ each element of which has a representative of zero entropy. There exists a finite set \mathcal{U} of pairwise disjoint closed annuli in $M \setminus X$ and an isomorphism $\phi : G \rightarrow \mathcal{G}$ where \mathcal{G} is a subgroup of $\text{Homeo}_0(M)$ such that*

1. *The elements of \mathcal{U} are permuted by \mathcal{G} and the restriction of each $g \in \mathcal{G}$ to an $A_i \in \mathcal{U}$ is a generalized Dehn twist.*
2. *For each $A_i \in \mathcal{U}$ there is some $g \in \mathcal{G}$ such that g leaves A_i invariant and $g|_{A_i}$ is an essential generalized Dehn twist. Conversely for any non-trivial element of G each of the annuli $A_j(g)$ guaranteed by Theorem 4.1 is isotopic to one of the $A_i \in \mathcal{U}$.*

3. The restriction of \mathcal{G} to $U = \bigcup_i A_i$ is a finitely generated subgroup of $\text{Homeo}_0(U)$.
The stabilizer in \mathcal{G} of any $A_i \in \mathcal{U}$ is infinite cyclic.

4. The restriction of \mathcal{G} to $M \setminus U$ is a finite group.

Proof. Let \mathcal{C} be the set of simple closed curves guaranteed by Lemma 4.5

We consider the set $\mathcal{U} = \{A_i\}$ of pairwise disjoint annular neighborhoods of representatives of each c_i and of each component of ∂M . We choose these so that $A_i \cap X = \emptyset$. By Theorem 4.1 we may choose a representative of each element of G which permutes the elements of \mathcal{U} and which is isotopic to a generalized Dehn twist on each A_i . Note that there may be some A_i and some g such that the restriction to A_i is inessential.

Let $U = \bigcup_i A_i$. By Theorem 4.1 elements of G may be represented by a finite set \mathcal{G}_0 of homeomorphisms of M whose each of whose restrictions to $M \setminus \text{int}(U)$ is a finite order isometry in a hyperbolic metric.

By Lemma 4.4 we may choose the representatives \mathcal{G}_0 so their restrictions to $M \setminus \text{int}(U)$ form a finite group of isometries of a hyperbolic metric. We may then choose \mathbb{T}^1 coordinates on $\bigcup_i \partial A_i$ so that the restriction of any element of \mathcal{G}_0 to ∂A_i is a rotation.

For each i we extend these coordinates on ∂A_i to $\mathbb{T}^1 \times I$ coordinates on A_i . For each $g_0 \in \mathcal{G}_0$ we observe that $g_0|_{A_i}$ is isotopic rel ∂A_i to a generalized Dehn twist from A_i to $g_0(A_i)$. Applying Lemma 3.2 we alter each g_0 by an isotopy supported on $\text{int}(U)$ to obtain a new homeomorphism $g : M \rightarrow M$ which is a generalized Dehn twist on each A_i . We denote by \mathcal{G} the set of elements obtained in this way from all the elements of \mathcal{G}_0 . It follows from the uniqueness and composition properties in Lemma 3.2 that elements of \mathcal{G} restricted to U form a group. Since the restrictions of \mathcal{G} and \mathcal{G}_0 coincide on $M \setminus \text{int}(U)$ we conclude that \mathcal{G} is a subgroup of $\text{Homeo}_0(M)$. Properties (1)-(4) are satisfied by construction. \square

Corollary 4.7. *Let M, X , and G be as in Lemma 4.6 and suppose H is a normal subgroup of G . Then H is finitely generated.*

Proof. Let \mathcal{U}, U and \mathcal{G} be as in Lemma 4.6, let \mathcal{H} be the subgroup of \mathcal{G} whose elements are representatives of elements of H and let \mathcal{K} be the (finite index) normal subgroup of \mathcal{G} which fixes each component of $M \setminus U$ pointwise. Then $\mathcal{H} \cap \mathcal{K}$ fixes each $U_i \in \mathcal{U}$ and restricts to Dehn twists on that U_i . It follows that $\mathcal{H} \cap \mathcal{K}$ is a finitely generated free abelian group. Since $\mathcal{H}/\mathcal{H} \cap \mathcal{K} \cong \mathcal{H}\mathcal{K}/\mathcal{K}$ and $\mathcal{H}\mathcal{K}/\mathcal{K}$ is subgroup of \mathcal{G}/\mathcal{K} which is finite we conclude \mathcal{H} is finitely generated. \square

5 The case of M with genus 0

We are now prepared to present the proof of Theorem 1.2.

Theorem 1.2 *Suppose M is an oriented compact connected surface of genus 0 and let G be a finitely generated infinite subgroup of $\mathcal{B}(M, X)$ with zero entropy, where X is a non-empty finite G -invariant set. Then if $\partial M \neq \emptyset$, or there is a point $x \in X$ fixed by all elements of G , then G is solvable. If $M = S^2$ then there is a finite index normal subgroup G_0 of G which is solvable and such that G/G_0 acts effectively on S^2 . Moreover when G is solvable the kernel K of the homomorphism $\pi : G \rightarrow \mathcal{S}_G(X)$ is free abelian.*

Proof. We may assume that the Euler characteristic of $M \setminus X$ is negative since otherwise X contains at most two points and $\mathcal{B}(M, X)$ is trivial or $\mathbb{Z}/2\mathbb{Z}$. Let \mathcal{U} be the set of annuli guaranteed by Lemma 4.6. The set \mathcal{U} must be non-empty since otherwise G would be finite.

Recall that U denotes the union of $A_i \in \mathcal{U}$ and let \mathcal{E} denote the components of the complement of U . The elements of \mathcal{E} are permuted by \mathcal{G} . We will prove the result by induction on $\text{Card}(\mathcal{E})$. If $\text{Card}(\mathcal{E}) = 1$ then each $A_i \in \mathcal{U}$ is a neighborhood of a boundary component of M and hence invariant under \mathcal{G} . It follows that $[\mathcal{G}, \mathcal{G}]$ acts trivially on U . By Lemma 4.6 $[\mathcal{G}, \mathcal{G}]$ acts as a finite abelian group on $M \setminus \text{int}(U)$ and it must then be trivial since it pointwise fixes the (non-empty) boundary. This provides the base case of our induction.

We form a graph Γ with one vertex for each element of \mathcal{E} and vertices joined by an edge if the corresponding elements of \mathcal{E} intersect a common annulus $A_i \in \mathcal{U}$. The graph Γ is a tree since a cycle in it would give rise to a curve in M which intersected an essential curve in some A_i in a single point.

Proposition 2.1 implies there is an element $E_0 \in \mathcal{E}$ which is preserved by the action of \mathcal{G} . Note this is trivial if there is $x \in X$ fixed by all elements of \mathcal{G} , because E_0 can be taken as the element of \mathcal{E} containing the global fixed point x .

Let \mathcal{G}_0 be the finite index normal subgroup of \mathcal{G} which preserves each component of ∂E_0 and let $\mathcal{G}_0^{(n)} = [\mathcal{G}_0^{(n-1)}, \mathcal{G}_0^{(n-1)}]$ for $n \geq 1$.

By Lemma 4.6 (4), we know that the action of \mathcal{G} restricted to E_0 is finite. This finite group (which we may assume are isometries) is isomorphic to $\mathcal{G}/\mathcal{G}_0$ because the action of G_0 on E_0 is trivial since G_0 preserves components of ∂E_0 and any finite order element of $\text{Homeo}_0(E_0)$ preserving ∂E_0 must be the identity. Note that this implies the group $\mathcal{G}/\mathcal{G}_0$ acts on E_0 and hence there is an effective action of the finite group $G/G_0 \cong \mathcal{G}/\mathcal{G}_0$ on S^2 . Also if there is a global fixed point $x \in X \cap E_0$ this finite group $\mathcal{G}/\mathcal{G}_0$ is abelian.

Let M_0 be the compact surface $M \setminus \text{int}(E_0)$. Let \bar{A} be the union of the A_i 's which intersect the boundary of E_0 . Then the restriction of \mathcal{G}_0 to the annuli \bar{A} leaves each annulus invariant and is abelian. Hence the action of $\mathcal{G}_0^{(1)} = [\mathcal{G}_0, \mathcal{G}_0]$ restricted to \bar{A} is trivial. Thus \mathcal{G}_1 preserves each of the components of M_0 . The number of components of the complement of reducing curves for each of these actions is smaller than the number of components for the action of \mathcal{G}_0 . Each of these actions is an action on the disk D^2 whose restriction to the boundary ∂D^2 has finite order. If we cone off the action on ∂D^2 we obtain an action of $\mathcal{G}_0^{(1)}$ with a global fixed point x_0 the cone

point. We now augment X by adding x_0 and note that by the induction hypothesis the restriction of $\mathcal{G}_0^{(1)}$ to a component of M_0 is solvable. It follows that $\mathcal{G}_0^{(1)}$ is solvable and consequently \mathcal{G}_0 is solvable. In the case that there is an $x \in X$ fixed by all elements of \mathcal{G} the group $\mathcal{G}/\mathcal{G}_0$ is abelian and hence $[\mathcal{G}, \mathcal{G}] \subset \mathcal{G}_0$ and \mathcal{G} is solvable. Since $G \cong \mathcal{G}$ we have completed the case that $M = S^2$.

We are left with the case that $\partial M \neq \emptyset$. In this case we cone the action of G on each boundary component of M . Since \mathcal{G} pointwise fixed ∂M all the disks added in the coning are fixed pointwise by G . We may therefore augment X by adding two points in each of the added disks to X to form X' . This guarantees that the natural map $B(M, X) \rightarrow B(S^2, X')$ is injective on G so applying the result for the sphere we obtain the result for a subgroup G of $B(M, X)$.

To obtain the fact that K is free abelian when G is solvable we note that elements of K fix X pointwise. Each element of K permutes elements of \mathcal{U} , but each such element bounds a disk containing a point of X . It follows that each element of K preserves each element of \mathcal{U} and is the identity on the complement of their union U . Since the restriction on each element of K to an element of \mathcal{U} is a Dehn twist, it follows that K is free abelian. \square

Corollary 1.3 *Suppose G satisfies the hypothesis of Theorem 1.2. If G is solvable then $G/K \cong \mathcal{S}_G(X)$. In particular the derived length satisfies*

$$\text{DLen}(G) - 1 \leq \text{DLen}(\mathcal{S}_G(X)) \leq \text{DLen}(G).$$

Proof. We note that $\mathcal{S}_G(X)$ is a homomorphic image of G so $\text{DLen}(\mathcal{S}_G(X)) \leq \text{DLen}(G)$. If $n = \text{DLen}(\mathcal{S}_G(X))$ it is clear that $G^{(n)} \subset K$. Hence since K is abelian $G^{(n+1)}$ is trivial, so $\text{DLen}(G) \leq n + 1$. Thus

$$\text{DLen}(G) - 1 \leq \text{DLen}(\mathcal{S}_G(X)) \leq \text{DLen}(G).$$

\square

6 The case of positive genus.

Let \mathcal{U} and \mathcal{G} be as in the conclusion of Lemma 4.6. Recall $U = \bigcup_i A_i$ and let \mathcal{E} be the components of $M \setminus \text{int}(U)$. We define \mathcal{U}_e to be the subset of \mathcal{U} consisting of those $A_i \in \mathcal{U}$ which are essential in M (not just in $M \setminus X$) and we let \mathcal{U}_c be $\mathcal{U} \setminus \mathcal{U}_e$, i.e., those elements of \mathcal{U} which are inessential in M . The core of an element of \mathcal{U}_c bounds a disk in M containing at least two points of X .

Lemma 6.1. *Suppose M has genus $g > 1$, and $\partial M = \emptyset$. Let G be a finitely generated subgroup of $\mathcal{B}(M, X)$ each of whose elements have zero entropy. Then*

1. If $\mathcal{U}_c \neq \emptyset$ and $\mathcal{U}_e = \emptyset$ then there is a normal subgroup of index $\leq 84(g-1)$ in \mathcal{G} which leaves invariant an element A of \mathcal{U} .
2. If $\mathcal{U}_e \neq \emptyset$ then there is a subgroup of \mathcal{G} which leaves invariant an element A of \mathcal{U} and whose index in \mathcal{G} is $\leq 84(g-1)(2g-2)!$.
3. If $\mathcal{U}_c = \mathcal{U}_e = \emptyset$ then \mathcal{G} is finite and has order at most $84(g-1)$.

Proof. If $\mathcal{U}_c \neq \emptyset$ and $\mathcal{U}_e = \emptyset$ then we let \mathcal{U}_m be the (non-empty) subset of maximal elements of \mathcal{U}_c under inclusion in disks. More precisely, $A \in \mathcal{U}_m$ if $A \in \mathcal{U}$ and one component of the complement of A is a disk in M and A is not a subset of an open disk which is a component of the complement of some other $A' \in \mathcal{U}$. There is only one component C of the complement of the union of all elements of \mathcal{U}_m which has positive genus (necessarily of genus g) since there are no elements of \mathcal{U}_m which are essential in M . The group \mathcal{G} must preserve the surface C and its action on C must be irreducible since $\mathcal{U}_e = \emptyset$. Hence the action of \mathcal{G} on C is finite. Since C has genus g the order of the restriction of \mathcal{G} to C is at most $84(g-1)$ (see Chapter 7 of [3]). It follows that the normal subgroup of \mathcal{G} which leaves invariant each of the elements of \mathcal{U}_m which intersects a boundary component of C has index at most $84(g-1)$. This completes (1).

If $\mathcal{U}_e \neq \emptyset$ then \mathcal{G} acts in a way that it permutes the elements of \mathcal{U}_e and permutes the components of the complement of their union. A component of the complement of \mathcal{U}_e may be an annulus containing points of X , but there must be at least one non-annular component. There are at most $2g-2 = |\chi(M)|$ non-annular components (see [5]) and they are permuted by \mathcal{G} . Hence there is a subgroup \mathcal{G}' of \mathcal{G} which preserves each of these components and which has index at most $(2g-2)!$ in \mathcal{G} . Any such non-annular component E has genus at most g . The action of \mathcal{G}' on E can be extended to \hat{E} which we define to be E with any boundary components coned off. Since there are no elements of \mathcal{U}_e in E , we may apply part (1) of this lemma to the group \mathcal{G}' acting on \hat{E} we conclude there is a normal subgroup \mathcal{G}'' of \mathcal{G}' of index at most $84(g-1)$ fixing at least one element of \mathcal{U} . This completes (2).

Finally if $\mathcal{U}_e = \mathcal{U}_c = \emptyset$ then $\mathcal{U} = \emptyset$ so \mathcal{G} is finite by Lemma 4.6 and has order at most $84(g-1)$ (see Chapter 7 of [3]).

□

Lemma 6.2. *Suppose M is an oriented compact connected surface of genus $g \geq 1$ and let G be a finitely generated subgroup of $\mathcal{B}(M, X)$ which contains no element with positive entropy and is realized by $\mathcal{G} \cong G$ as in Lemma 4.6. If any of the following hold:*

1. $\partial M \neq \emptyset$, or
2. $\text{Fix}(\mathcal{G}) \cap X \neq \emptyset$, or
3. there exists an $A \in \mathcal{U}$ which is \mathcal{G} -invariant,

then the group G is solvable.

Proof. We induct on the genus g of M and the cardinality of \mathcal{U} . More precisely given M and X we will show the result holds if it holds for surfaces of lower genus and for surfaces of the same genus with a lower cardinality of \mathcal{U} . We first address the base cases of the induction. If the genus $g = 0$ this is provided by Theorem 1.2. If $\mathcal{U} = \emptyset$ Lemma 4.6 implies that \mathcal{G} is finite and we consider the three possible hypotheses. If $\partial M \neq \emptyset$ and \mathcal{G} fixes it pointwise then \mathcal{G} is trivial. If $x \in \text{Fix}(\mathcal{G}) \cap X$ then \mathcal{G} acts in an abelian fashion on a neighborhood of x and hence on all of M . If there exists an $A \in \mathcal{U}$ which is \mathcal{G} -invariant, then the restriction of $G^{(1)} = [G, G]$ to A is trivial and hence $G^{(1)}$ acts trivially on M . Hence in all cases G is solvable. Thus the result holds if the genus g is zero or the cardinality of \mathcal{U} is zero.

We proceed to the inductive step, first in the case that $\partial M \neq \emptyset$. Let B be a component of ∂M . The boundary component B may be a boundary component of an annulus $A \in \mathcal{U}$ (which collars B). Let E_1 be the component of $\mathcal{E} = M \setminus U$ which intersects B or an annulus $A \in \mathcal{U}$ which contains B . The group $\mathcal{G}^{(1)} = [\mathcal{G}, \mathcal{G}]$ acts as the identity on B and hence on one component of the boundary of E_1 . Since every element of $\mathcal{G}^{(1)}$ acts as an irreducible map on E_1 and fixes one component of its boundary pointwise, $\mathcal{G}^{(1)}$ must act trivially on E_1 . If we consider the components of the complement of E_1 the group $\mathcal{G}^{(1)}$ preserves them and each of them has either lower genus or contains only a proper subset of \mathcal{U}_c or both. The induction hypothesis then implies that $\mathcal{G}^{(1)}$ restricted to these components is solvable. This completes the proof part (1), namely that \mathcal{G} is solvable when $\partial M \neq \emptyset$.

If there exists $x \in X$ fixed by each element of \mathcal{G} then we may blow it up to obtain a surface M' with boundary on which \mathcal{G} acts. Since x lies in the interior of the complement of U the action of \mathcal{G} on $\partial M'$ is abelian. The previous case applied to M' then shows that $\mathcal{G}^{(1)}$ is solvable so \mathcal{G} is also.

We now consider the case that $\partial M = \emptyset$. Let \mathcal{U} and \mathcal{G} be as in the conclusion of Lemma 4.6 and let \mathcal{G}_e and \mathcal{U}_e be as in Lemma 6.1. By hypothesis there is an $A \in \mathcal{U}$ which is \mathcal{G} invariant. The group $\mathcal{G}^{(1)}$ acts trivially on A . Let $M'' = M \setminus \text{int}(A)$ so each component of M'' is $\mathcal{G}^{(1)}$ invariant and $\mathcal{G}^{(1)}$ fixes the components of $\partial M''$ pointwise. $\mathcal{G}^{(1)}$ acts on each component of M'' fixing the boundary pointwise and hence this action is solvable by part (1) of this theorem. \square

We can now complete the proofs of Theorem 1.6 and Theorem 1.7. Recall that in both of these results X is a non-empty finite set.

Theorem 1.6 *Suppose M is an oriented compact connected surface of genus $g > 1$ and let G be a finitely generated infinite subgroup of $\mathcal{B}(M, X)$ which contains no element with positive entropy.*

1. *If $\partial M \neq \emptyset$, or there exists $x \in X$ fixed by each element of G , then G is solvable.*
2. *Otherwise G contains a solvable subgroup G_0 of index at most $84(g-1)(2g-2)!$ in G .*

Proof. If $\partial M \neq \emptyset$ then the result follows from Lemma 6.2. If $\partial M = \emptyset$ and $\mathcal{U}_c \neq \emptyset$ then by part (1) of Lemma 6.1 there is a subgroup of index $\leq 84(g-1)$ in \mathcal{G} which leaves invariant an element A of \mathcal{U} . This subgroup is solvable by Lemma 6.2.

If $\partial M = \emptyset$ and $\mathcal{U}_c = \emptyset$ and $\mathcal{U}_e \neq \emptyset$ then by part (2) of Lemma 6.1 there is a subgroup of index at most $84(g-1)(2g-2)!$ in \mathcal{G} which leaves invariant an element A of \mathcal{U} . This subgroup is solvable by Lemma 6.2.

Finally, if $\mathcal{U}_e = \mathcal{U}_c = \emptyset$ then part (3) of Lemma 6.1 implies \mathcal{G} is finite contradicting our hypothesis. \square

Only the proof of Theorem 1.7 remains.

Theorem 1.7 *Suppose M is an oriented compact connected surface of genus $g = 1$ and let G be a finitely generated subgroup of $\mathcal{B}(M, X)$ which contains no element with positive entropy. Then G is solvable.*

Proof. If $\partial M \neq \emptyset$ the result follows from Lemma 6.2, so we may assume $\partial M = \emptyset$ and $M = \mathbb{T}^2$. As before let \mathcal{U}_e denote the subset of elements of \mathcal{U} which are essential in M and let \mathcal{G}_e be the subgroup of \mathcal{G} which preserves each element of \mathcal{U}_e . Then all elements of \mathcal{U}_e must be parallel in M and separated by points of X . We may choose a simple closed curve γ in $M \setminus X$ crossing each $U_i \in \mathcal{U}_e$ exactly once. A circular order on γ induces a circular order on the components of the complement of annuli in \mathcal{U}_e or equivalently on the elements of \mathcal{U}_e . This order must be preserved by elements of \mathcal{G} . It follows that the action of \mathcal{G} on \mathcal{U}_e factors through a finite cyclic group and that $\mathcal{G}/\mathcal{G}_e$ is a finite cyclic group. We conclude that $\mathcal{G}^{(1)} = [\mathcal{G}, \mathcal{G}]$ preserves all elements of \mathcal{U}_e . We may therefore apply Lemma 6.2 to $\mathcal{G}^{(1)}$ to obtain the desired result. \square

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